Mathematical Modeling

A tutorial

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Network Theory and Computer Modeling in the Study of Religion
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A note on myself

June 1997: two exams on the same day

- János Kertész: Computer simulations (physics major)
- József Schweitzer: Jewish liturgy (Hebrew major)

Anything common in these two topics?

HOPEFULLY...

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Theses on

- Analysis of DNA sequences using text analysis methods (physics major, supervisor: Tamás Vicsek)
- Modeling Mathematically the Statistical Properties of Written Texts (theoretical linguistics major)

From physics to linguistics: was it a big step?

NO!

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Overview

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Let us create a calculating machine

The machine has to be able to sum up (two) numbers.

Input:	Tamás Biró
Output:	István Czachesz
Programmer:	Luther Martin
Processing units:	everybody else

Only rule type allowed for each processing unit:

if	*you hear X_1*	*[and X_2*	*[and $X_3 \ldots$]],*
then	*say Y_1 to Z_1*	*[and say Y_2 to Z_2*	*[and Y_3 to Z_3 [...]]*

20 minutes for the project!

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What have we learned?

- Computation $\neq$ computers!
- Seemingly intelligent processes can be automated.
- Computational resources: memory (# of processing units) and time.
- Human resources: the time to create the program.
- Need to precisely define everything. Bugs and debugging.
- Continuous time *vs.* discrete time ticks.
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David Marr: Three levels of analysis

1. **Computational level:** What does the system do? What is the function (i.e., mapping input onto output) performed by the system? E.g., summation; face recognition; ritual performance.

2. **Algorithmic/representational level:** How is it performed? Representations, and manipulations of those representation. E.g., summation digit-by-digit.

3. **Implementational/physical level:** How is this algorithm physically realized? E.g., in silico; wetware; workshop participants.

Overview

Methodologies in physics

A. Experimental physics: data collection
(exploratory research *vs.* hypothesis testing)

B. Theoretical physics: mathematics for modeling the word/nature

+ [Thought experiments]

+ Computer simulations (e.g., Kertész and Vicsek)

Analogy in other disciplines?

"Complicatedness" of theories

0. **Thought experiments:** handled mentally.
1. **Mathematical models:** handled analytically.
2. **Computer simulations:** can be more complex than mathematically tractable models, but simpler than real life.

Are we happy with

- *Level of abstraction?*
- *Simplifications?*

3. **Experiments:** complexities of real life controlled.
4. **Observations:** complexities of real life at their best.

Numerical solution *vs.* analytic solution

$$1 + 2 + 3 + \ldots + 98 + 99 + 100 = ?$$

- **Numerical solution:** go and compute it with sheer force.
  For more complex problems: often an approximate solution, only.
- **Analytic solution:** clever math provides a closed formula.
  Exact solution with pencil and paper
  $\rightarrow$ on the condition that an analytic solution exists!

$$1 + 2 + \ldots + 99 + 100 = \frac{100 \times (100 + 1)}{2} = 50 \times 101 = 5050$$

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Issues with computer simulations

- Helps you better understand your theory/hypothesis.
- Forces you to formulate details of theory/hypothesis precisely.
- Faster. Can also be applied to past/remote/unreal conditions. Etc.
- Level of optimal abstractions:
  - If too simple: no connection to reality? *What do the results tell us?*
  - If too complex, too many parameters: easy to tweak the model. *What do the results tell us?*

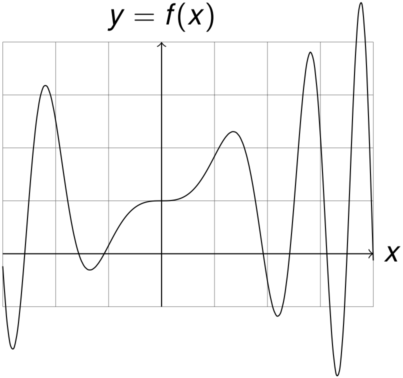
$\rightarrow$ Possible answer:

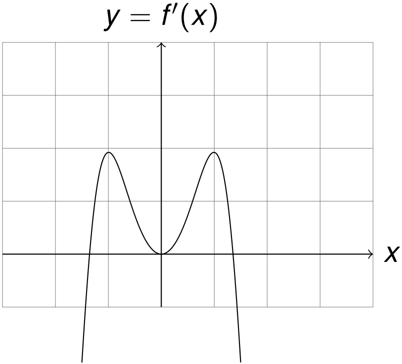
- Understand the behavior of the model as a function of its parameters.
- Seek results that are not too dependent on parameter setting.

Overview

Derivatives

https://en.wikipedia.org/wiki/Derivative#/media/File:Tangent_function_animation.gif





$$f(x) = x\sin(x^2) + 1 \quad \Rightarrow \quad f'(x) = \sin(x^2) + 2x^2\cos(x^2)$$

Derivatives

$f(x)$	$f'(x)$
c	0
x	1
x^2	$2x$
x^3	$3x^2$
$c \cdot f(x)$	$c \cdot f'(x)$
$f(x) + g(x)$	$f'(x) + g'(x)$
e^x	e^x
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
...	...

Differential equations

What is $f(x)$, if

$$f'(x) = 2x$$

Solution:

$$\begin{aligned} f(x) &= x^2 \\ f(x) &= x^2 + c \end{aligned}$$

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Differential equations

What is $f(x)$, if

$$f''(x) = -f(x)$$

Solution:

$$\begin{aligned} f(x) &= \sin(x) \\ f(x) &= \cos(x) \\ f(x) &= c_1 \cdot \sin(x) + c_2 \cdot \cos(x) \end{aligned}$$

Thus far: analytic solutions

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Numerical solutions for differential equations

What is $f(x)$, if

$$f''(x) + x \cdot f'(x) - 2 \cdot x^2 \cdot f(x) + \cos(x^3) - 15 = 0$$

Solution:

Use computers to solve this problem.
Numerical solutions: e.g., using step-by-step approximations.
NB: various sources of errors.

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Population dynamics

$y(t)$: size of the population at time t.

$\Delta y(t) = y(t+1) - y(t)$: population growth at time t.

Suppose that population growth is equal to population size:

$$\begin{aligned} y(t+1) - y(t) &= y(t) \\ y(t+1) &= 2y(t) \end{aligned}$$

Then: $y(1) = 2y(0)$, $y(2) = 2y(1) = 4y(0)$,
$y(3) = 2y(2) = 8y(0), \ldots, y(t) = 2^t y(0)$.

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Suppose that population growth is equal to population size:

$$\begin{aligned} y(t+1) - y(t) &= y(t) \\ \Delta y(t) &= y(t) \\ \frac{\mathrm{d}y}{\mathrm{dt}} = y'(t) &= y(t) \end{aligned}$$

And so: $y(t) = e^t$.

Dynamic system

$1, 2, \ldots n$: the components of the dynamics system.

$y_1(t), y_2(t), \ldots y_n(t)$:
"value" of each component in the dynamics system at time t.

The equations defining the dynamic system **(discrete time!)**:

$$
\begin{aligned}
y_1(t+1) &= \ldots y_1(t) + \ldots y_2(t) + \ldots y_n(t) + \ldots t + \ldots \\
y_2(t+1) &= \ldots y_1(t) + \ldots y_2(t) + \ldots y_n(t) + \ldots t + \ldots \\
\ldots & \\
y_n(t+1) &= \ldots y_1(t) + \ldots y_2(t) + \ldots y_n(t) + \ldots t + \ldots
\end{aligned}
$$

So what functions are $y_1(t), y_2(t), \ldots y_n(t)$?
Solve those differential equations either numerically, or analytically.

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The equations defining the dynamic system **(continuous time!)**:

$$\begin{aligned}
y_1'(t) &= \ldots y_1(t) + \ldots y_2(t) + \ldots y_n(t) + \ldots t + \ldots \\
y_2'(t) &= \ldots y_1(t) + \ldots y_2(t) + \ldots y_n(t) + \ldots t + \ldots \\
\ldots \\
y_n'(t) &= \ldots y_1(t) + \ldots y_2(t) + \ldots y_n(t) + \ldots t + \ldots
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Solve those differential equations either numerically, or analytically.

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Conclusions

And now: I am expected to provide smart conclusions!

But what if

- you gave them
- or postpone them to the end of the day/end of the week?

Anyway. . .

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Anyway. . .

Thank you for your attention!

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